# Counterexamples in percolation: the site percolation critical probabilities $p_H$ and $p_T$ are unequal for a class of fully triangulated graphs



1984 J. Phys. A: Math. Gen. 17 637

(http://iopscience.iop.org/0305-4470/17/3/025)

J. Phys. A: Math. Gen. 17 (1984) 637–646. Printed in Great Britain

# Counterexamples in percolation: the site percolation critical probabilities $p_H$ and $p_T$ are unequal for a class of fully triangulated graphs

John C Wierman

Department of Mathematical Sciences, Johns Hopkins University, Baltimore, Maryland 21218, USA

Received 12 July 1983

**Abstract.** A family of fully triangulated graphs is given, for which the critical percolation probabilities $p_T$ and $p_H$ are unequal, and the clusters-per-site function considered by Sykes and Essam is a polynomial. The graphs are modifications of an example of Van den Berg.

## 1. Introduction

Since percolation models were introduced by Broadbent and Hammersley (1957), a fundamental problem has been the determination of critical probabilities for various graphs. An early breakthrough on this problem was the non-rigorous method of determining exact critical probabilities proposed by Sykes and Essam (1964). The Sykes and Essam approach produced conjectured values of $\frac{1}{2}$ for the square lattice bond model and triangular lattice site model, $2\sin(\pi/18)$ for the triangular lattice bond model, and $1-2\sin(\pi/18)$ for the hexagonal lattice bond model. The Sykes and Essam paper stimulated mathematical research in percolation theory which led to the rigorous verification of these values by Kesten (1980, 1982) and Wierman (1981), following significant developments in the theory by Seymour and Welsh (1978) and Russo (1978). For discussion of the proofs of these results, the reader may consult the survey by Wierman (1982) or the monograph by Kesten (1982).

The Sykes and Essam method is based on the limiting open clusters-per-site function for a graph. For each graph, this function is assumed to have a unique singularity, denoted by $p_E$, which is interpreted as the critical probability. By deriving a relationship between the clusters-per-site functions of matching graphs using Euler's law, and either the self-matching property of a graph or a transformation between a graph and its matching graph, Sykes and Essam determined the values given above without explicitly determining the clusters-per-site functions for the graphs. The function is not explicitly known for any of these graphs. Also, it is unknown whether a singularity actually exists or is unique. For the square lattice bond model, Kesten (1981) has shown that the clusters-per-site function is analytic except at $p=\frac{1}{2}$, and that it has two continuous derivatives everywhere in $[0, 1]$. In this paper, a family of graphs is presented for which the clusters-per-site function may be explicitly determined, and is found to have

† Research supported by the National Science Foundation under Grant Nos MCS-8118229 and MCS-8303238.

0305-4470/84/030637+10\$02.25  

no singularity. Note that the term 'singularity' does not necessarily refer to a point of non-analyticity. However, this is the usual interpretation, with no well defined alternative interpretation developed in the literature.

Sykes and Essam also implicitly assume that there is a unique critical probability value, or, in other words, that all alternative interpretations of the critical probability or percolation threshold concept yield the common value $p_E$. The most common alternative interpretations of the critical probability concept are based on the probability of existence of infinite open clusters and on expected size of open clusters. To define these critical probabilities, consider site percolation on a graph $G$ in which each site is open with probability $p$, $0 \leq p \leq 1$, independently of all other sites. Let $P_p$ denote the corresponding probability measure, and $E_p$ denote the corresponding expectation operator. The open cluster containing a site $s$, denoted $W_s$, is the maximal connected subgraph of open sites in $G$ which contains $s$. The infinite cluster size critical probability is defined by

$$p_H = \inf\{p: P_p(W_s = \infty) > 0\}.$$

For a connected graph $G$, the value of $p_H$ is independent of the choice of site $s$. The mean cluster size critical probability is defined by

$$p_T = \inf\{p: E_p(W_s) = \infty\},$$

which is also independent of the choice of $s$ if $G$ is connected. For technical reasons, Seymour and Welsh (1978) defined the sponge-crossing critical probability as the threshold above which the probabilities of existence of a path of open sites that connects opposite sides of an increasing sequence of rectangles in the graph converge to a positive limit. The sponge-crossing critical probability, denoted $p_S$, proved to be the key to the rigorous determinations of critical probabilities mentioned above, where it was proved that $p_H = p_T = p_S$. The relationship of the Sykes and Essam definition to the other three critical probabilities is unclear, however.

One conclusion of the method of Sykes and Essam is that the critical probability of the site percolation model on any fully triangulated planar graph is $\frac{1}{2}$. However, Van den Berg (1981) constructed a fully triangulated planar graph for which the site percolation model has critical probability $p_H = p_T = 1$. This example is still consistent with the belief that all critical probabilities must be equal for fully triangulated planar graphs, however. In this paper, by modifying Van den Berg's example, graphs are constructed for which the critical probabilities may be explicitly determined, and which exhibit more unusual behaviour. The graphs constructed as counterexamples are not regular, and may be considered 'non-physical'. However, while regular lattices are more commonly considered for percolation models, the Sykes and Essam claims were not made only for regular graphs, and have been asserted for non-regular graphs. For each $x$ in $(0, 1)$, there exists a fully triangulated graph $G_x$ with $p_T = x$ and $p_H = 1$ for the site percolation model. Thus the two critical probabilities are not equal, and in fact are related only by the trivial inequality that for any graph $p_T \leq p_H$. Note also that, since the bond model on $G_x$ is easily seen to have $p_H = 1$, it is possible for a bond percolation critical probability ($p_H$) to be strictly greater than a site percolation critical probability ($p_T$) for the same graph. For this class of graphs it is also possible to determine the limiting mean clusters-per-site function that is the basis for the Sykes and Essam method of determining critical probabilities. The function is actually a polynomial. Thus there is no singularity of the function, so the critical probability $p_E$

is not well defined. Graphs satisfying these properties are constructed in § 2. The clusters-per-site function is evaluated in § 3.

From any planar graph $G$ which is not fully triangulated, a fully triangulated graph $G'$ may be constructed by inserting one or more diagonals in each non-triangular face. By the inclusion principle, the critical probability of $G$ must be at least as large as the critical probability of $G'$. Thus, the Sykes and Essam conjecture that the critical probability of a fully triangulated graph is $\frac{1}{2}$ implies that the critical probability of every planar graph (without multiple edges) is at least $\frac{1}{2}$. Van den Berg's example and the examples in § 2 do not disprove this conjecture for the critical probability $p_H$. However, a different modification of Van den Berg's example produces a graph with critical probability $p_H$ at most $1/(k-1)$, for each positive integer $k$. This class of graphs is discussed in § 4.

## 2. Unequal critical probabilities

### *2.1. Van den Berg's example*

A graph $G^0$, shown in figure 1, is constructed as a nested sequence of triangles $T_1, T_2, T_3, \ldots$ with alternating orientations. For odd integers $n$, the triangle $T_n$ points upward, while for even integers $n$, $T_n$ points downward. To provide a convenient origin, insert a site $T_0$ in the centre of $T_1$, and insert bonds that connect $T_0$ to each of the three vertices of $T_1$. For each $n \geq 1$, label the sites of $T_n$ counter-clockwise from the positive $x$ axis as $t_{n,1}$, $t_{n,2}$, and $t_{n,3}$.

Though $G^0$ is not fully triangulated, a fully triangulated graph $G^{\mathrm{v}}$ may be constructed by inserting a bond $b_{n,i}$ connecting $t_{n,i}$ to $t_{n+2,i}$ for each $n \geq 1$ and $i = 1, 2, 3$ (see figure 2). The graph $G^{\mathrm{v}}$ was constructed by Van den Berg (1981) as a counterexample to the claim by Sykes and Essam that site models on fully triangulated graphs have critical probability $\frac{1}{2}$. For $G^{\mathrm{v}}$, $p_H = p_T = 1$, since the largest integer $n$ such that $T_n$ is reached by an open path from the origin is bounded by $2X+1$, where $X = \inf\{k \geq 1$: $t_{n,i}$ is open for all $n = 2k, 2k+1$ and $i = 1, 2, 3\}$ is a geometrically distributed (parameter $(1-p)^6$) random variable and is thus finite almost surely (AS) if $p < 1$.

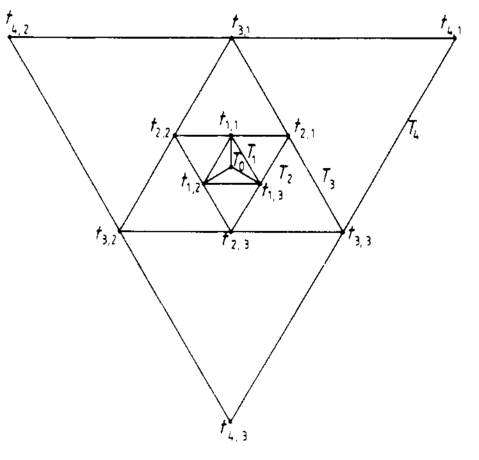


**Figure 1.**

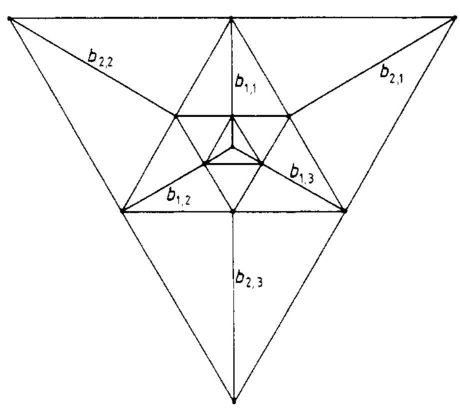


**Figure 2.**

### 2.2. *The class of graphs*

For each $x \in (0, 1)$, we now construct a fully triangulated graph $G_x$ from $G^{\vee}$. On each bond $b_{n,i}$, insert $[x^{-n}]$ equally spaced sites labeled $b_{n,i,j}$, $1 \leq j \leq [x^{-n}]$ as $b_{n,i}$ is travelled from $t_{n,i}$ to $t_{n+2,i}$. ($[\cdot]$ denotes the greatest integer function.) Insert bonds connecting $b_{n,i,j}$ to $t_{n,i}$ and $t_{n,i-1(\mathrm{mod}\ 3)}$ for each $n$, $i$ and $j$ (see figure 3).

The additional sites and bonds were inserted in such a manner that the expected cluster size is increased greatly, but the probability of existence of an infinite cluster is relatively unchanged. For each $G_x$, $0 < x < 1$, the site model critical probability $p_H$ is 1, by the same reasoning used for $G^{\vee}$. This reasoning also applies to the bond model to obtain $p_H = 1$, by considering possible barriers of 12 bonds containing the boundaries of alternate triangles in the original sequence.

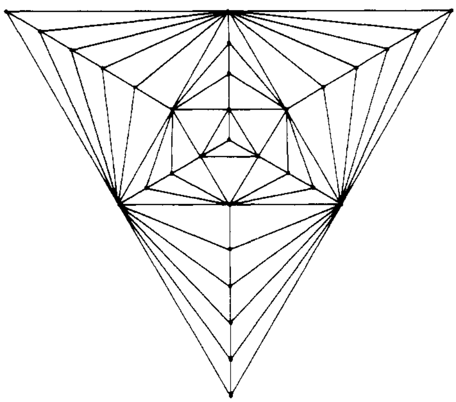

**Figure 3.**

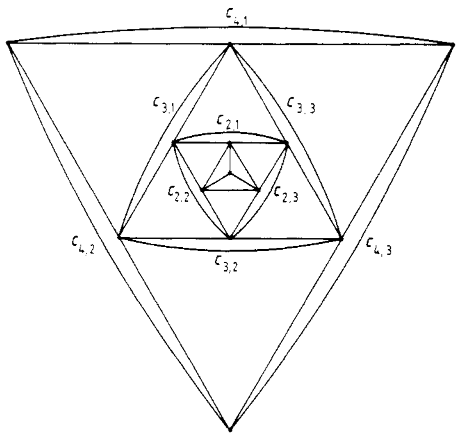


**Figure 4.**

### 2.3. *Reduction to a simpler graph*

As a first step in evaluating the critical probability $p_T$ for $G_x$ we will derive probability bounds for a simpler graph which has percolative behaviour similar to that of $G_x$.

For fluid to pass through a line segment $b_{n,i}$ in $G_x$, all $[x^{-n}]$ sites on the segment must be open. Since this event has probability $p^{[x^{-n}]}$, which converges to zero rapidly as $n \to \infty$, we will neglect this possibility in constructing an approximating graph. Likewise, if the sites $t_{n,i}$ and $t_{n,i+(-1)^n(\mathrm{mod}\ 3)}$ are both open, then fluid from one will wet the other if any site on $b_{n-1,i}$ is open. The probability of this event is $1-(1-p)^{[x^{-n}]}$, which converges rapidly to 1 as $n \to \infty$. To approximate this behaviour, consider the two sites to be connected by a bond.

Therefore we will study the approximating graph $G^{\wedge}$, shown in figure 4, which is constructed by deleting all bonds $b_{n,i}$ from $G^{\vee}$, and inserting bonds $C_{n,i}$ connecting $t_{n,i}$ and $t_{n,i+1(\mathrm{mod}\ 3)}$, $i = 1, 2, 3$, for each $n \geq 2$. $G^{\wedge}$ provides a good approximation to the percolative behaviour of $G_x$ when considering probabilities of open connections between sites far from the origin.

### 2.4. *Probability bounds for $G^{\wedge}$*

Note that if one site in $T_n$ is wetted from the origin in $G^{\wedge}$, then any other site of $T_n$ that is open is wetted directly from it through one of the bonds $c_{n,i}$, $i = 1, 2, 3$. Therefore each site that is wetted from the origin in $G^{\mathrm{A}}$ is wetted by fluid flowing along a path which does not backtrack toward the origin, i.e. does not pass through a site of $T_{n-1}$ after it has passed through a site of $T_n$. Whether a site of $T_n$ is wetted in $G^{\mathrm{A}}$ depends only on which sites of $T_n$ are open and which sites of $T_{n-1}$ are wetted from the origin in $G^{\mathrm{A}}$. Thus the probability that a site in $T_n$ is wetted from the origin in $G^{\mathrm{A}}$ may be computed by a Markov chain approach.

Let $X_n = 0$, 1, or 2, as the number of sites in $T_n$ which are wetted is 0, 1, or 2 or more respectively. $\{X_n\}$, $n \geqslant 0$, is a Markov chain with an absorbing state 0. The cases where two or three sites of $T_n$ are wetted may be combined into one state, since in either case all three sites of $T_{n+1}$ are adjacent to wetted sites, so the transition probability vectors are equal. The transition matrix for the Markov chain $\{X_n\}$ is given by

$$P = \begin{bmatrix} 1 & 0 & 0 \\ q^2 & 2pq^2 & p^3+3p^2q \\ q^3 & 3pq^2 & p^3+3p^2q \end{bmatrix}.$$

Since the absorbing state corresponds to the eigenvalue of 1, by diagonalising $P$ to compute its powers easily (e.g. see Karlin and Taylor (1975), p 541) one sees that the exponential rate of decay of the probability that $T_n$ is wetted is determined by the second largest eigenvalue, which is

$$f(p) = \tfrac{1}{2}(p^3+3p^2q) + \tfrac{1}{2}(p^6+6p^5q+9p^4q^2+4p^2q^4)^{1/2},$$

where $q = 1-p$. Clearly $f$ is a continuous function of $p$, and one may check that $f$ is strictly increasing from 0 to 1. Thus there exist constants $K$ and $k$, independent of $n$, for which

$$kf(p)^n \leqslant P^n(i,j) \leqslant Kf(p)^n$$

for all $i, j = 1, 2$.

### *2.5. Probability bounds for $G_x$*

We first obtain an upper bound for the probability that $T_n$ is wetted in $G_x$. Note that inserting the bonds $c_{n,i}$ in $G_x$ creates a non-planar graph, denoted $G_x^{\mathrm{A}}$. Since $G_x$ is a subgraph of $G_x^{\mathrm{A}}$,

$$P(T_{n+m} \text{ wet in } G_x | T_n \text{ wet}) \leqslant P(T_{n+m} \text{ wet in } G_x^{\mathrm{A}} | T_n \text{ wet}).$$

Then

$$\begin{aligned} &P(T_{n+m} \text{ wet in } G_n^{\mathrm{A}} \text{ wet}) \\ &\qquad \leqslant P(T_{n+m} \text{ wet in } G^{\mathrm{A}} | T_n \text{ wet}) + P(\text{all sites on } b_{k,i} \text{ are open for some} \\ &\qquad\qquad n \leqslant k \leqslant n+m-1, i = 1,2,3) \\ &\qquad \leqslant Kf(p)^m + 3mp^{[x^{-n}]} \\ &\qquad < 2Kf(p)^m \end{aligned}$$

for $n \geqslant N(m)$ sufficiently large. Therefore

$$\begin{aligned} &P(T_n \text{ wet in } G_x) \\ &\qquad \leqslant P(T_n \text{ wet in } G_x^{\mathrm{A}}) \end{aligned}$$

$$= P(T_0 \text{ open}) \prod_{k=1}^{[n/m]} P(T_{(k+1)m} \text{ wet in } G_x^{\mathrm{A}} | T_{km} \text{ wet in } G_x^{\mathrm{A}})$$

$$= P(T_0 \text{ open}) \prod_{k=1}^{[N(m)/m]} P(T_{(k+1)m} \text{ wet in } G_x^{\mathrm{A}} | T_{km} \text{ wet in } G_x^{\mathrm{A}})$$

$$\times \prod_{k=N(m)/m}^{[n/m]} P(T_{(k+1)m} \text{ wet in } G_x^{\mathrm{A}} | T_{km} \text{ wet in } G_x^{\mathrm{A}})$$

$$\leqslant C_1(p, m)(2Kf(p)^m)^{(n-N(m))/m+1}$$

$$= C_2(p, m)K^{n/m}f(p)^n$$

for $n \geqslant N(m)$, where $C_1$ and $C_2$ are independent of $n$.

Follow a similar procedure to obtain a lower bound. Note that by deleting all the bonds on $b_{n,i}$ connecting sites $b_{n,i,j}$, a new graph $G_x^{\mathrm{D}}$ is created (see figure 5). The connection between two open sites of $T_n$ which is certain in $G_x^{\mathrm{A}}$ fails in $G_x^{\mathrm{D}}$ with probability $(1-p)^{[x^{-n}]}$. Thus

$$P(T_n \text{ wet in } G_x) \geqslant P(T_n \text{ wet in } G_x^{\mathrm{D}}).$$

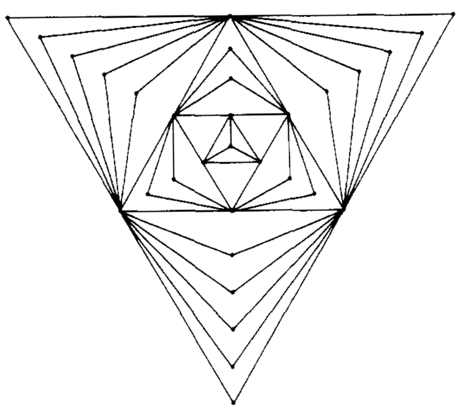

**Figure 5.** $G_x^{\mathrm{D}}$.

To calculate a lower bound for $G_x^{\mathrm{D}}$, we use

$$P(T_{n+m} \text{ wet in } G_x^{\mathrm{D}} | T_n \text{ wet})$$

$$\geqslant P(T_{n+m} \text{ wet in } G_x^{\mathrm{A}} | T_n \text{ wet}) - P(\text{all sites on } b_{k,i} \text{ are closed for some } n \leqslant k \leqslant n+m-1, i = 1, 2, 3),$$

$$\geqslant kf(p)^m - 3m(1-p)^{[x^{-n}]}$$

$$\geqslant \tfrac{1}{2}kf(p)^m$$

for $n \geqslant M(m)$ sufficiently large.

Continuing as before, we obtain

$$P(T_n \text{ wet in } G_x) \geqslant C_3(p, m)k^{n/m}f(p)^n$$

for $n \geqslant M(m)$ where $C_3$ is independent of $n$.

*2.6. Evaluation of $p_T$*

Note that if a site of $T_n$ is wet then there are at least $2[x^{-n}]$ sites and at most $3[x^{-n}]$ sites of $b_{n,i}$, $i=1,2,3$, which are adjacent to a wet site of $T_n$. Therefore

$$2p[x^{-n}]+1 \leqslant E(W_0 \cap \{T_n \cup \{b_{n,i}: i=1,2,3\}\} | T_n \text{ wet}) \leqslant 3p[x^{-n}]+3,$$

since each $b_{n,i,j}$ adjacent to a wet site of $T_n$ is in the open cluster $W_0$ with probability $p$.

Multiplying by $P(T_n \text{ wet})$, using the bound above, and summing over $n$, we obtain

$$E[W_0] \geqslant C_4(p,m) + C_3(p,m) \sum_{n=M(m)}^{\infty} k^{n/m} f(p)^n (2p[x^{-n}]+1)$$

where $C_4(p,m)$ represents the expected number of sites in the open cluster $W_0$ inside the triangle with vertices $T_{M(m)}$. Since this series diverges when $k^{1/m}f(p)/x \geqslant 1$, we have that $p \geqslant f^{-1}(x/k^{1/m})$ implies $p \geqslant p_T$. Thus $f^{-1}(x/k^{1/m}) \geqslant p_T$ for any positive integer $m$, so $f^{-1}(x) \geqslant p_T$.

Similarly,

$$E[W_0] \leqslant C_5(p,m) + C_2(p,m) \sum_{n=N(m)}^{\infty} K^{n/m} f(p)^n (3p[x^{-n}]+3)$$

which converges when $K^{1/m}f(p)/x < 1$. Therefore, $f^{-1}(x/K^{1/m}) \leqslant p_T$ for any positive integer $m$, so $f^{-1}(x) \leqslant p_T$.

Thus, $p_T = f^{-1}(x)$ for $G_x$. Since $f^{-1}$ is continuous and strictly increasing from 0 to 1, for every $y \in (0,1)$ there exists a graph, $G_{f(y)}$, which has $p_T = y$ and $p_H = 1$.

## 3. Expected clusters-per-site function

We now evaluate the limiting open clusters-per-site function for the graphs $G_x$, $0<x<1$. In the following, computations are given for the expected clusters-per-site functions only for regions bounded by the triangles with vertex sets $\{T_n\}$. This is done for simplicity, with the same limit obtained for any increasing sequence of triangles centred at the origin or any other fixed point, or for an increasing sequence of similar rectangles centred at a common point.

Consider a line segment $b_{n,i}$ in $G_x$. If either site $t_{n+1,i}$ or $t_{n+1,i+(-1)^n (\text{mod } 3)}$ is open, then all open sites $b_{n,i,j}$ on $b_{n,i}$ are in a single open cluster on $G_x$ which contains a site of $T_{n+1}$. However, if both $t_{n+1,i}$ and $t_{n+1,i+(-1)^n (\text{mod } 3)}$ are closed, each run of successive open sites on $b_{n,i}$ is in a distinct open cluster.

To count the open clusters in the region bounded by the triangle with vertices $T_n$, we count separately the clusters which contain a vertex of $T_k$ for some $k \leqslant n$, and the clusters which contain no vertex of $T_k$ for any $k$, which are clusters formed by runs of open sites in $b_{k,i}$, $1 \leqslant k \leqslant n$, $i=1,2,3$.

Note that there are $3n+1$ sites in $\bigcup_{i=0}^{n} T_n$, and thus at most $3n+1$ open clusters containing such sites. Since there are

$$\sum_{k=1}^{n-2} 3[x^{-k}] + 3n + 1$$

sites in the region, these clusters do not contribute to the limiting clusters-per-site function. The limiting clusters-per-site function is thus determined entirely by the runs on the line segments $b_{k,i}$, $1 \leq k \leq n-1$, $i = 1, 2, 3$.

To count open clusters on $b_{k,i}$, define inductively an alternating sequence of random variables by

$$O_1 = \inf\{j\colon b_{k,i,j} \text{ is open}\}, \qquad C_1 = \inf\{j > O_1\colon b_{k,i,j} \text{ is closed}\}$$

and for $m \geq 2$,

$$O_m = \inf\{j > C_{m-1}\colon b_{k,i,j} \text{ is open}\}, \qquad C_m = \inf\{j > O_m\colon b_{k,i,j} \text{ is closed}\}.$$

Note that $C_i - O_i$ are IID random variables with a geometric $(1-p)$ distribution, and $O_{i+1} - C_i$ are IID random variables, independent of $\{C_i - O_i\}$ also, with a geometric $(p)$ distribution. Therefore $O_{i+1} - O_i$ are IID with mean $1/(1-p) + 1/p = 1/p(1-p)$, so by the strong law of large numbers,

$$\frac{O_m}{m} = \frac{\sum_{i=2}^{m}(O_i - O_{i-1})}{m} + \frac{O_1}{m} \xrightarrow{\text{AS}} \frac{1}{p(1-p)}.$$

Defining $T_r = \sup\{m\colon O_m \leq r\}$, we have

$$\frac{T_r}{T_r + 1}\,\frac{T_r + 1}{O_{T_r} + 1} \leq \frac{T_r}{r} \leq \frac{T_r}{O_{T_r}}.$$

Applying the almost sure convergence of $O_m/m$ along the subsequence $T_r$, we find that

$$T_r/r \to p(1-p) \qquad \text{almost surely,}$$

so by the dominated convergence theorem,

$$E[T_r/r] \to p(1-p) \qquad \text{as } r \to \infty.$$

Letting $R_{k,i}$ denote the number of runs of open sites on $b_{k,i}$ which do not contain $t_{k,i}$ or $t_{k+2,i}$, we have

$$T_{[x^{-k}]} - 2 \leq R_{k,i} \leq T_{[x^{-k}]},$$

so

$$E(R_{k,i})/[x^{-k}] \to p(1-p).$$

If exactly two of the sites $t_{k+1,i}$, $i = 1, 2, 3$, are closed, then one of the line segments $b_{k,i}$ may contribute open clusters which do not include vertices of $T_{k+1}$. If all three sites $t_{k+1,i}$, $i = 1, 2, 3$, are closed, then all three $b_{k,i}$, $i = 1, 2, 3$, contribute open clusters.

The expected number of clusters which contain no sites of $T_k$, $1 \leq k \leq n$, is then

$$\sum_{k=1}^{n-2} [E(R_{k,1})p(1-p)^2 + 3E(R_{k,1})(1-p)^3].$$

Divide by the total number of sites in the region, and let $n \to \infty$ to obtain the limiting clusters per site function for $G_x$

$$\tfrac{1}{3}[p^2(1-p)^2 + 3p(1-p)^4].$$

Since the function is a polynomial, there is no singularity, so the Sykes and Essam critical probability $p_E$ is not well defined.

Note also that the limiting clusters per site function is identical for all graphs $G_x$, $0 < x < 1$, so a cluster-per-site function does not uniquely identify a graph.

## 4. Example with $p_H < \frac{1}{2}$

We now construct a class of fully triangulated graphs $G_k$, $k \geq 2$, such that $p_H \leq 1/(k-1)$ for $G_k$ for all $k \geq 2$. This class provides a counterexample to the conjecture that $p_H \geq \frac{1}{2}$ for all fully triangulated graphs.

To construct $G_k$, begin with the graph $G^{\mathrm{v}}$. For $n \geq 2$, label the bonds on the perimeter of the triangle with vertices $T_n$ as $d_{n,i}$, $1 \leq i \leq 6$, moving counterclockwise from $t_{n,1}$ (see figure 6). Insert $k^{n-2}$ equally spaced sites on $d_{n,i}$ for all $n$ and $1 \leq i \leq 6$. For each $i = 1, 2, 3$, insert bonds connecting the site $t_{1,i}$ to each of the $2k$ sites on $d_{3,2i-1}$ and $d_{3,2i}$.

The remaining non-triangular faces are all partitioned into triangles by the same procedure. Fix $n \geq 3$ and $1 \leq i \leq 6$. Label the sites inserted on $d_{n,1}$ as $e_1, e_2, \ldots, e_{k^{n-3}}$ and the sites inserted on $d_{n+1,1}$ as $f_1, f_2, \ldots, f_{k^{n-2}}$, starting in both cases from the vertex in $T_n$ where $d_{n,i}$ and $d_{n+1,i}$ intersect. For each $j = 1, 2, \ldots, k^{n-3} - 1$, insert bonds connecting $e_j$ to $f_{(k-1)j+1}, \ldots, f_{kj+1}$. Also insert bonds connecting $e_{k^{n-3}}$ to $f_{k^{n-2}-k+1}, \ldots, f_{k^{n-2}}$, and the site at the endpoint of $d_{n,i+1}$ in $T_{n+1}$ (see figure 7 for $G_3$).

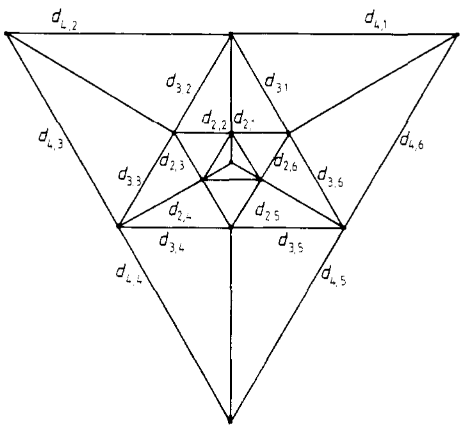


**Figure 6.**

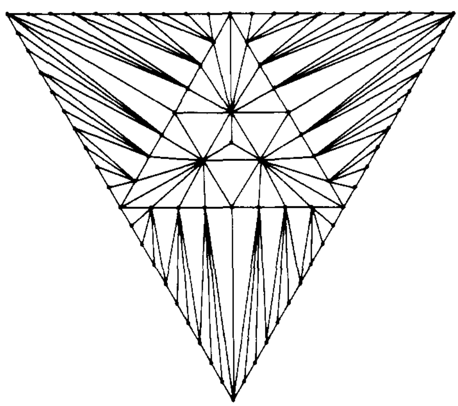

**Figure 7.** $G_3$.

The graph $G_k$ contains a Bethe tree with coordination number $k$, as may be seen by deleting all bonds of $G^{\mathrm{v}}$ and the bonds from $e_j$ to $f_{kj+1}$, $1 \leq j \leq k^{n-3} - 1$, and from $e_{k^{n-3}}$ to $T_{n+1}$ in the construction above. Such a Bethe tree has critical probability $p_H = p_T = 1/(k-1)$ (see e.g. Essam (1972)). Thus, by its inclusion in $G_k$, the critical probability $p_H \leq 1/(k-1)$ for $G_k$.

## References

Broadbent S R and Hammersley J M 1957 *Proc. Camb. Phil. Soc.* **53** 629–41, 642–5
Essam J W 1972 *Phase Transitions and Critical Phenomena* vol 2 (New York: Academic) pp 197–270
Karlin S and Taylor H M 1975 *A First Course in Stochastic Processes* (New York: Academic)
Kesten H 1980 *Commun. Math. Phys.* **74** 41–59
—— 1981 *J. Stat. Phys.* **25** 717–56
—— 1982 *Percolation Theory for Mathematicians* (Boston: Birkhauser)

Russo L 1978 *Z. Wahrsch. verw. Geb.* **43** 39–48
Seymour P D and Welsh D J A 1978 *Ann. Discrete Math.* **3** 227-45
Sykes M F and Essam J W 1964 *J. Math. Phys.* **5** 1117–27
Van den Berg J 1981 *J. Math. Phys.* **22** 152–7
Wierman J C 1981 *Adv. Appl. Prob.* **13** 293–313
—— 1982 *Ann. Prob.* **10** 509–24